

Finding Small Solutions to Low Degree Polynomials and Applications

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Plan

Finding Small Solutions

Applications to RSA

Conclusion

Goal

Theorem

Let N be an integer and $f \in \mathbb{Z}_N[x]$ monic, $\deg f = d$.

Then we can efficiently find all

$$x \in \mathbb{Z} : |x| \leq B \text{ and } f(x) \equiv 0 \pmod{N}$$

for $B = N^{1/d}$.

Main Idea

Finding roots over \mathbb{Z}_N ?

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Let's find $g \in \mathbb{Z}[x]$, such that

$$f(x) \equiv 0 \pmod{N} \implies g(x) = 0.$$

Main Idea

How? We want to prevent overflowing N :

Let $x \in \mathbb{Z}, |x| < B$.

If $|g(x)| < N$, then

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$$g(x) = x^d + a_{d-1}x^{d-1} + \dots + a_1x + a_0 \in \mathbb{Z}[x].$$

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We want for all i and for all x with $|x| < B$:

$$|a_i x^i| < \frac{N}{d+1} \quad \left(\quad |a_i| < \frac{1}{B^i} \frac{N}{d+1} \right).$$

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How to find a "good" multiple? Use LLL!

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The last column corresponds to $f(x)$.

The other columns correspond
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a short vector $v \in L$:

$$\begin{pmatrix} ka_0 \pmod N \\ ka_1 \pmod N \\ \vdots \\ ka_{d-1} \pmod N \\ k \pmod N \end{pmatrix}_{(d+1) \times 1}$$

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$$\begin{pmatrix} ka_0 \pmod N \\ (ka_1 \pmod N)B \\ \vdots \\ (ka_{d-1} \pmod N)B^{d-1} \\ (k \pmod N)B^d \end{pmatrix}$$

Recall that we want $B^j a_j^0 < \frac{N}{d+1}$ (a_j^0 is a coefficient in new polynomial).

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Scale coordinates!) minimize $B^i a_i^0 = (ka_i \pmod N)B^i$.

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Upper-triangular structure:

$$\det(L) = N \cdot BN \cdots B^{d-1}N \cdot B^d = N^d B^{d(d+1)/2}.$$

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$$, \quad B < \frac{N^{\frac{2}{d(d+1)}}}{2^{(d+1)^{2/d}}} = O(N^{\frac{2}{d(d+1)}}).$$

Idea 2: Variable/Polynomial Multiples

$$f(x) = 0 \pmod{N} \quad \Rightarrow \quad h(x)f(x) = 0 \pmod{N},$$

for any $h \in \mathbb{Z}_N[x]$.

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Careful: increases degree!

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$$\begin{array}{l}
 1 \\
 x \\
 x^2 \\
 \vdots \\
 x^{d-1} \\
 x^d \\
 \vdots \\
 x^{2d-1}
 \end{array}
 \begin{array}{c}
 x^0 \quad x^1 \quad \quad \quad x^{d-1} \quad f(x) \quad x \cdot f(x) \quad \quad \quad x^{d-1} \cdot f(x) \\
 \hline
 \left(\begin{array}{cccccccc}
 N & 0 & 0 & 0 & a_0 & 0 & \dots & 0 \\
 0 & BN & \ddots & \vdots & \vdots & Ba_0 & \ddots & \vdots \\
 \vdots & 0 & \ddots & 0 & \vdots & \vdots & \ddots & 0 \\
 \vdots & \vdots & \ddots & B^{d-1}N & B^{d-1}a_{d-1} & \vdots & \vdots & B^{d-1}a_0 \\
 \vdots & \vdots & \ddots & 0 & B^d & B^d a_{d-1} & \vdots & \vdots \\
 \vdots & \vdots & \ddots & \ddots & \ddots & B^{d+1} & \ddots & \vdots \\
 \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & B^{2d-2}a_{d-1} \\
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 x^{d-1} & \vdots & \vdots & 0 & B^d & B^d a_{d-1} & \vdots & \vdots \\
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$$\det(L) = N \cdot BN \cdot \dots \cdot B^{d-1}N \cdot B^d \cdot B^{d+1} \cdot \dots \cdot B^{2d-1} = N^d B^{d(2d-1)}.$$

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Idea 3: f -Multiples (mod N^m)

Idea 1: $L = \{f(x)g \mid \{N, Nx, Nx^2, \dots, Nx^{d-1}\}\}.$

Idea 2: $L = \dots \mid \{f(x), xf(x), \dots, x^{d-1}f(x)\}$

(increase degree of the polynomials).

Idea 3: $L = ???$

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Idea 3: increase degree of N :

consider polynomials mod N^m .

powers of $f(x)$ allow to lift:

$L = \{N^{m-i} f(x)^i x^j \mid 0 \leq i < m, 0 \leq j < d-1\}$.

covers Ideas 1 and 2!

Idea 3: f -Multiples (mod N^m)Example: $d = 2, m = 2$

$$\begin{array}{c}
 \\
 x^0 \\
 x^1 \\
 x^2 \\
 x^3 \\
 x^4 \\
 x^5
 \end{array}
 \begin{pmatrix}
 N^2 x^0 & N^2 x^1 & N^1 x^0 f(x)^1 & N^1 x^1 f(x)^1 & x^0 f(x)^2 & x^1 f(x)^2 \\
 N^2 & ? & ? & ? & ? & ? \\
 0 & N^2 B^1 & ? & ? & ? & ? \\
 0 & 0 & N^1 B^2 & ? & ? & ? \\
 0 & 0 & 0 & N^1 B^3 & ? & ? \\
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 0 & 0 & 0 & N^1 B^3 & ? & ? \\
 0 & 0 & 0 & 0 & B^4 & ? \\
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 \end{pmatrix}$$

Same upper-triangular structure) easy calculation of $\det(L)$.

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$$\dim(L) = d(m + 1),$$

$$\det(L) = N^{dm(m+1)/2} B^{(d(m+1) - 1)d(m+1)/2}.$$

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$$B < \alpha(d, m) N^{\frac{m}{d(m+1)-1}} = \alpha(d, m) N^{\frac{1}{d}} \Theta\left(\frac{1}{m}\right).$$

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RSA - Recap

- p, q two (large) primes, private
- $n = p \cdot q$, public
- exponents: e public, d private such that

$$ed \equiv 1 \pmod{\text{lcm}(p-1, q-1)}$$

- encryption: $c = m^e \pmod n$
- decryption: $m = c^d \pmod n$

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- Assume small e , e.g. $e = 3$.
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Example

- Let $e = 3$, $N = 1000003$.
- Let $m = 100$, then $c = m^e \pmod N = 1000000$.
- Clearly, $m = \sqrt[3]{1000000} = 100$.

“Cube” attack (Stereotyped messages)

- Assume small e , e.g. $e = 3$.
- Assume m is close to a constant α : $m = \alpha + m_0$, $m_0 < N^{1/e}$.
- Example: constant padding:
“today’s secret password is: **llattice**”.

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- More generally, $c = L(m_0)^e \pmod N$, where $L_i \in \mathbb{Z}_{N_i}[x]$ is a public affine map.

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“today’s secret password is: **Illattice**”.
- More generally, $c = L(m_0)^e \pmod N$, where $L_i \in \mathbb{Z}_{N_i}[x]$ is a public affine map.
- Coppersmith: $L(m_0)^e$ is a degree- e polynomial, $m_0 < N^{\frac{1}{e}}$ is a small root!

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- Let $m = 1234567 + 7777777m_0$, where $m_0 = 50$.

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- We know that

$$m^e - c = 892450m_0^3 + 1866122m_0^2 + 726335m_0 + 302637 = 0 \pmod N,$$

$$m_0^3 + 1684527m_0^2 + 1652432m_0 + 1942344 = 0 \pmod N.$$

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$$m_0^3 + 1684527m_0^2 + 1652432m_0 + 1942344 \equiv 0 \pmod N.$$
- Use Coppersmith's method, get $m_0 = 50$.

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- CRT: reconstruct C such that $C = m^e \pmod{N_1 N_2 \dots N_e}$.

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- Note that $m < N_1 \Rightarrow m^e < N_1 N_2 \dots N_e$.

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- **Broadcasting scenario:**
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- $c_1 = m^e \pmod{N_1}, \dots, c_e = m^e \pmod{N_e}$.
- CRT: reconstruct C such that $C = m^e \pmod{N_1 N_2 \dots N_e}$.
- Note that $m < N_1 \Rightarrow m^e < N_1 N_2 \dots N_e$.
- Again: $m = \sqrt[e]{C}$, over \mathbb{Z} .

Hastad Broadcast Attack - Harder

- Assume small e , e.g. $e = 3$.
- Unrestricted m .
- Protected broadcasting:
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- Can we break this?

Hastad Broadcast Attack - Harder

- Let $g_i(x) := (L_i(x)^e - c_i) \pmod{2 \mathbb{Z}_{N_i}[x]}$.
- Note $g_i(m_0) = 0 \pmod{N_i}$.

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- In particular, $g(m_0) = 0 \pmod{N_1 N_2 \dots N_e}$.

Hastad Broadcast Attack - Harder

- Let $g_i(x) := (L_i(x)^e - c_i) \pmod{2} \in \mathbb{Z}_{N_i}[x]$.
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- How? Simply apply CRT to the coefficients.

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-) recover m_0 !

```
sage.rings.polynomial.polynomial_modn_dense_ntl.small_roots(self, X=None, beta=1.0, epsilon=None,
**kwds)
```

Let N be the characteristic of the base ring this polynomial is defined over: $N = \text{self.base_ring().characteristic()}$. This method returns small roots of this polynomial modulo some factor b of N with the constraint that $b \geq N^\beta$. Small in this context means that if x is a root of f modulo b then $|x| < X$. This X is either provided by the user or the maximum X is chosen such that this algorithm terminates in polynomial time. If X is chosen automatically it is $X = \text{ceil}(1/2N^{\beta^2/\delta-\epsilon})$. The algorithm may also return some roots which are larger than X . 'This algorithm' in this context means Coppersmith's algorithm for finding small roots using the LLL algorithm. The implementation of this algorithm follows Alexander May's PhD thesis referenced below.

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- x – an absolute bound for the root (default: see above)
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- epsilon – the parameter ϵ described above. (Default: $\beta/8$)
- ****kwds** – passed through to method `Matrix_integer_dense.LLL()`.

Note on Sagemath!

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3. you get $X = \lceil N^{1/d} \cdot 1/8 / 2 \rceil$.
4. for example, $d = 3$) $X = \lceil N^{5/24} / 2 \rceil$,
instead of "expected" $N^{1/3} = N^{8/24}$!
5. need to compute required ϵ manually before calling...

```
1  from sage.all import *
2
3  N = next_prime(10**50)
4  E = 3
5  x = PolynomialRing(Zmod(N), names='x').gen()
6
7  m0 = 10**12 + 20190607 # secret
8  X = 2*10**12 # bound
9
10 m = 1234567890 * m0 + 11223344556677889900
11 c = pow(m, E, N)
12
13 poly = (1234567890 * x + 11223344556677889900) ** E - c
14 poly /= poly.leading_coefficient()
15
16 epsilon = RR(1/poly.degree() - log(2*X, N))
17 if epsilon <= 0:
18     print "Too large bound X!"
19     quit()
20
21 print "epsilon:", "2~%f" % RR(log(epsilon, 2))
22 for root in poly.small_roots(epsilon=epsilon):
23     print "root", root
```

Plan

Finding Small Solutions

Applications to RSA

Conclusion

A good resource by David Wong:

github.com/mimoo/RSA-and-LLL-attacks

- implementation of univariate and bivariate Coppersmith algorithms in Sage (from scratch, using LLL);
- also a survey on lattice-based attacks with a good intro.

Another good resource:

[Alexander May's Dissertation](#)