Attacks on the Legendre PRF

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joint work with
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Plan

Legendre Symbol, PRG, PRF

Let p be an odd prime.

$$\left(\frac{a}{p}\right) = \begin{cases} 1, & \text{if } a = b^2 \text{ for some } b \neq 0, \\ 0, & \text{if } a = 0, \\ -1, & \textit{otherwise}. \end{cases}$$

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Damgård (CRYPTO 1988), *conjecture*:

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Grassi *et al.* (CCS 2016):

$$\left(\tfrac{k+x}{p}\right) \text{ is a very efficient } \textit{PRF} \text{ for MPC.}$$

Cryptanalysis of the PRF

Ref.	Time	#Queries	Comment
Initial claim (CCS 16)	$O(p)$	$O(\log p)$	exhaustive search
Khovratovich (ia.cr/2019/862)	$\tilde{O}(\sqrt{p})$	$\tilde{O}(\sqrt{p})$	birthday-bound
Beyne *et al.* (ia.cr/2019/1357)	$\tilde{O}(\sqrt{p})$	$O(\sqrt[4]{p})$	reduced data complexity

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Notation

$$\ell(a) := \left\lfloor \frac{1}{2}\left(1 - \left(\frac{a}{p}\right)\right) \right\rfloor$$ *0 - quadratic residue, 1 - non-residue*

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Assumption (heuristic):
$L_k([m])$ has few collisions when $m = \Omega(\log p)$.

Table-based Attack

Let $m = \lceil \log p \rceil$

Step 1: Fill Table $\mathcal{T}_k$

query and store

$$L_k(a_i + [m])$$

for Q/m
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A collision (a_i, b_j) reveals the key: $k = b_j - a_i$.

Complexity: Time: $O(Q + p \log^2 p/Q)$,
Memory: $O(Q)$ bits

Reducing Queries (1)

Let's exploit *multiplicativity* of the Legendre symbol:

$$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right) \quad \text{for all } a, b \in \mathbb{F}_p.$$

In terms of ℓ:

$$\ell(ab) = \ell(a) \oplus \ell(b) \quad \text{for all } a, b \in \mathbb{F}_p^*,$$

Reducing Queries (2)

Consider a sequence $(a, a+1, a+2, \ldots) \subseteq \mathbb{F}_p$.

$a+0 \quad a+1 \quad a+2 \quad a+3 \quad a+4 \quad a+5 \quad a+6 \quad a+7$

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$$||$$

$$\tfrac{a}{2}+0 \quad \tfrac{a+1}{2}+0 \quad \tfrac{a}{2}+1 \quad \tfrac{a+1}{2}+1 \quad \tfrac{a}{2}+2 \quad \tfrac{a+1}{2}+2 \quad \tfrac{a}{2}+3 \quad \tfrac{a+1}{2}+3$$

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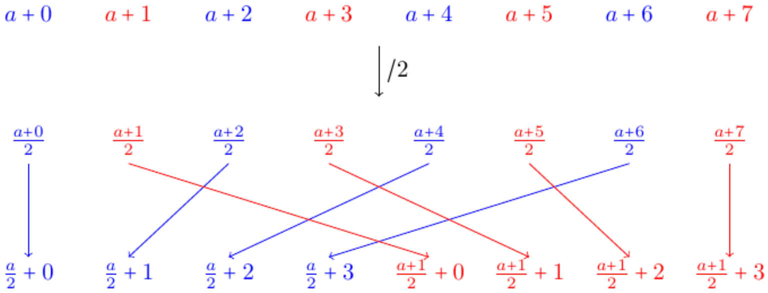
Consider a sequence $(a, a+1, a+2, \ldots) \subseteq \mathbb{F}_p$.



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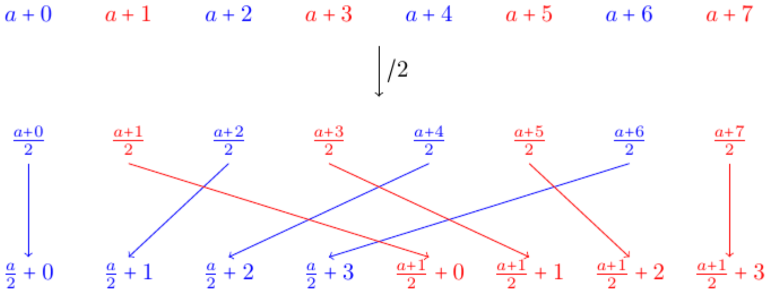

Two new sequences starting at $\frac{a}{2}$ and $\frac{a+1}{2}$!

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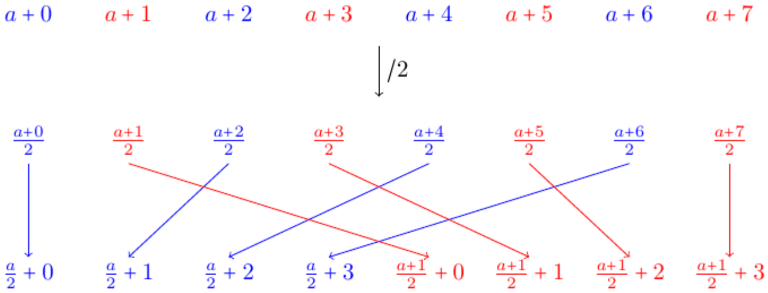

Legendre symbols ℓ match up to the constant $\ell(2)$.

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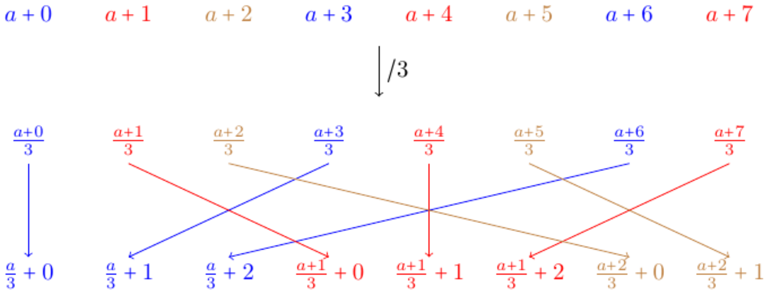

Can use any other (small) number instead of 2,
as long as sequence length allows.

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Open Problem: better data structure/algorithm for subsequence lookup?

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Example for $d = 3$:

$$L_{x^3+k_2x^2+k_1x+k_0}(a) = \ell(a^3 + k_2a^2 + k_1a + k_0),$$
$$\text{secret: } k_0, k_1, k_2 \in \mathbb{F}_p$$

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- Weak key attacks!

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$$\tilde{v} := \big(\ell(f(a_1)), \ell(f(a_2)), \ldots, \ell(f(a_m))\big) \in \mathbb{F}_2^m$$

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3. Compute all p vectors with $r \in \mathbb{F}_p$:

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- Factors of $f(x)$ do not have to be linear.
- Birthday-bound attack when there is a factor of degree $\lfloor d/2 \rfloor$.

The End

Open Problems

1. Better data structure/algorithm for subsequence lookup?
2. Birthday-bound attacks for all keys? (Higher-degree L.PRF)
3. Beyond-birthday-bound attacks?

ia.cr/2019/1357