Revisiting Meet-in-the-Middle Cryptanalysis of SIDH/SIKE with Application to the $IKEp182 Challenge

Aleksei Udovenko[1,2], Giuseppe Vitto[2]

[1]CryptoExperts
[2]SnT, University of Luxembourg

Selected Areas in Cryptography 2022
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CRYPTO EXPERTS

High-level Overview

- SIDH/SIKE are isogeny-based PQ protocols
- Rely on hardness of finding *isogenies* between elliptic curves
- (Previously) Best attacks: generic claw finding (meet-in-the-middle)
- Physical memory constraints (size x speed)
  $\Rightarrow$ low-memory van Oorschot-Wiener (vOW)

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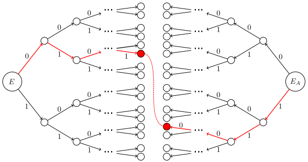

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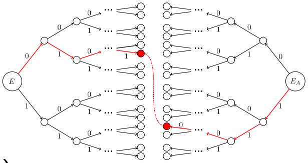

- **this work**: revisiting and optimizing the MitM approach
- Proof-of-concept: breaking \$IKEp182 challenge (by Microsoft) ~~on a laptop on a weekend~~ on an HPC cluster in a week (9 core-years)

Comparison to the Castryck-Decru Attack (Castryck and Decru 2022)

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Outdated? Is MitM useless?

- Castryck-Decru attack relies on the torsion point images
- MitM is more generally applicable (existing + future schemes)
- Generic attack in the generic setting may still be relevant for security analysis

Plan

SIDH/SIKE

- Public parameters:
  1. prime p with $p+1 = 2^{e_A}3^{e_B}$
  2. starting curve E with 2^{e_A}- and 3^{e_B}-torsion bases

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Figure credits: TikZ for Cryptographers

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This work: recovering the 2^{e_A}-isogeny $\phi_A : E \to E/\langle A\rangle$, given only E and $E_A = E/\langle A\rangle$

SIDH/SIKE Arithmetic

- Montgomery curves: $E_A : y^2 = x^3 + Ax^2 + x$ defined over $\mathbb{F}_{p^2}$
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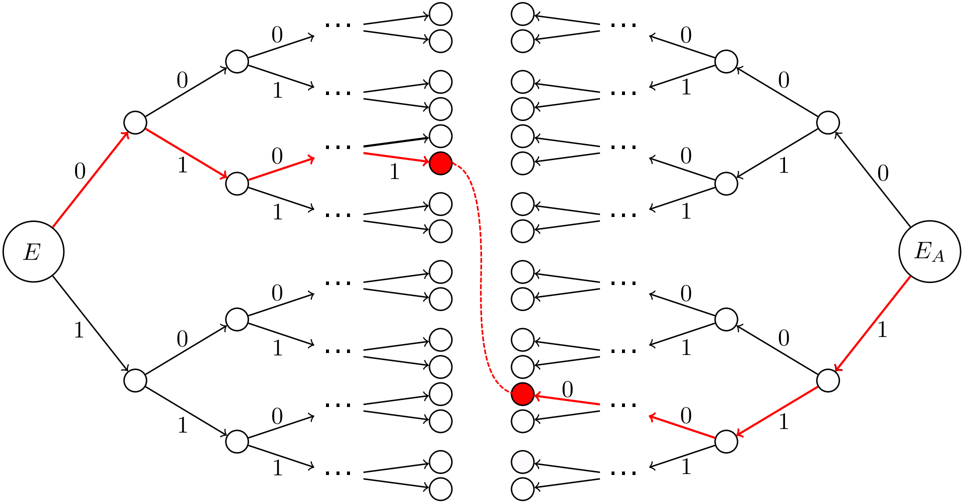
- The 2-kernels can be derived from the 2^{e_A}-kernel of ϕ_{Alice} by pushing through ϕ_i and raising to appropriate power $[2^{e_A-1-i}]$

Plan

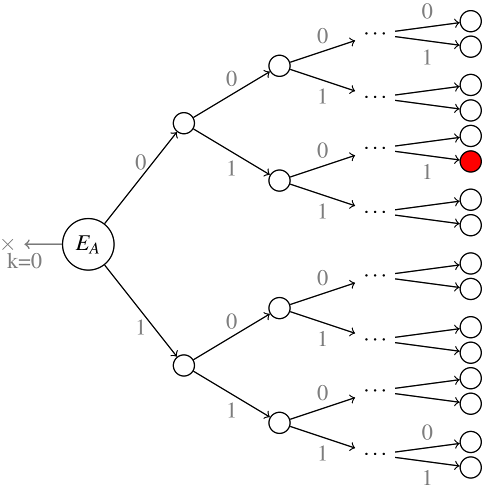
High-level MitM (Galbraith 1999; Adj et al. 2019)



SIKE: Left Tree

- Goal:

$$\text{LeftTree} = \left\{ j(E_A/\langle P + [s]Q \rangle) \mid s \in [0, 2^{e_A/2}] \right\}$$



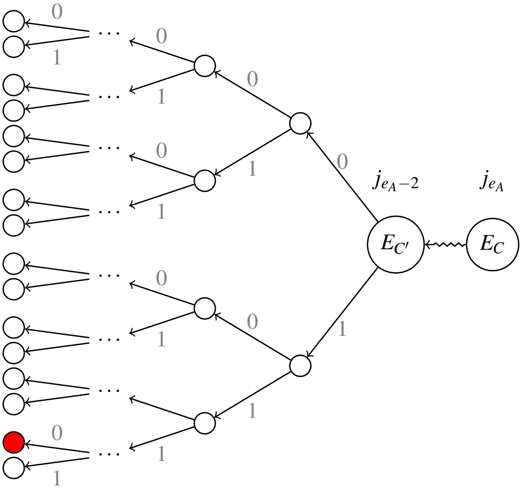
SIKE: Right Tree

- Optimized arithmetic formulas leak 2 last steps (Costello et al. 2020)
- We express the right tree in the same shape as the left tree
- Let

$$C' = 2\frac{C-6}{C+2}$$

- Goal:

$$\mathrm{RightTree} = \left\{ j(E_{C'}/\langle P' + [s]Q' \rangle) \mid s \in [0, 2^{e_A/2}] \right\}$$



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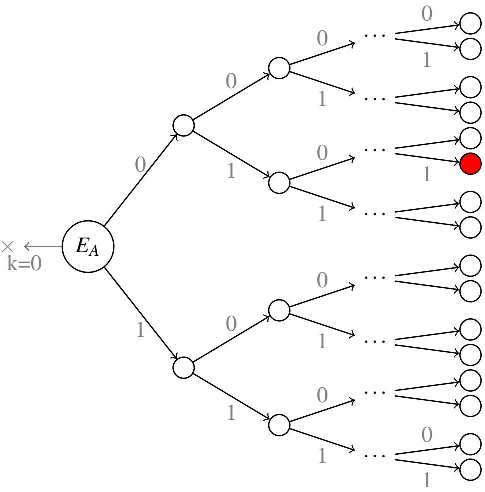
Recursive Generation (MitM-DFS (Adj et al. 2019))

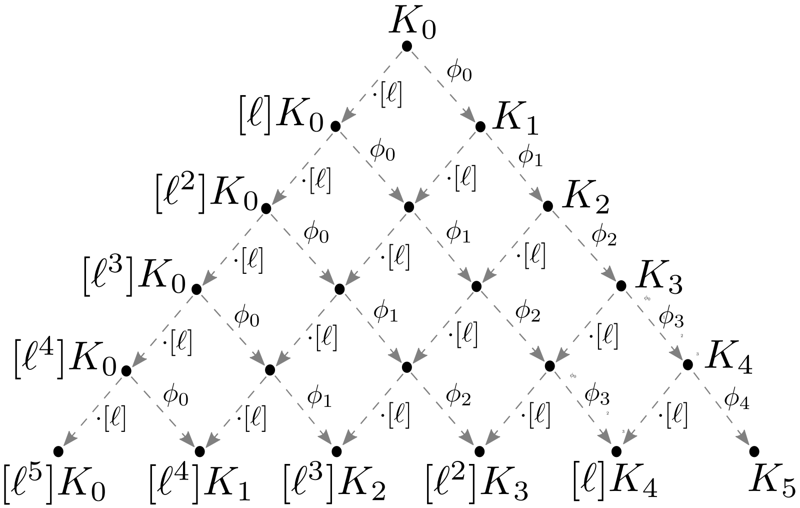
- Goal:

$$\text{Tree} = \left\{ j(E/\langle P + [s]Q\rangle) \mid s \in [0, 2^{e_A/2}] \right\}$$

- DFS visit tree (recursively)
  - need 2-kernel points
  - maintain $2^{e_A - i}$-torsion, by pushing it through isogenies and updating accordingly
- We adapt the idea to SIKE's optimized arithmetic (e.g., ensure $x \neq 0$)
- We adapt optimal strategy for trade-off between doublings and isogeny evaluations



Optimal Strategy for Doubling-Isogeny Trade-off



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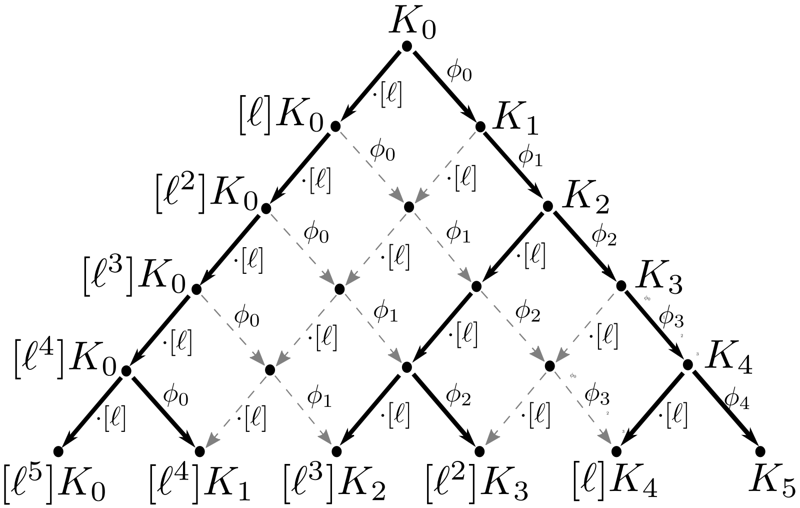

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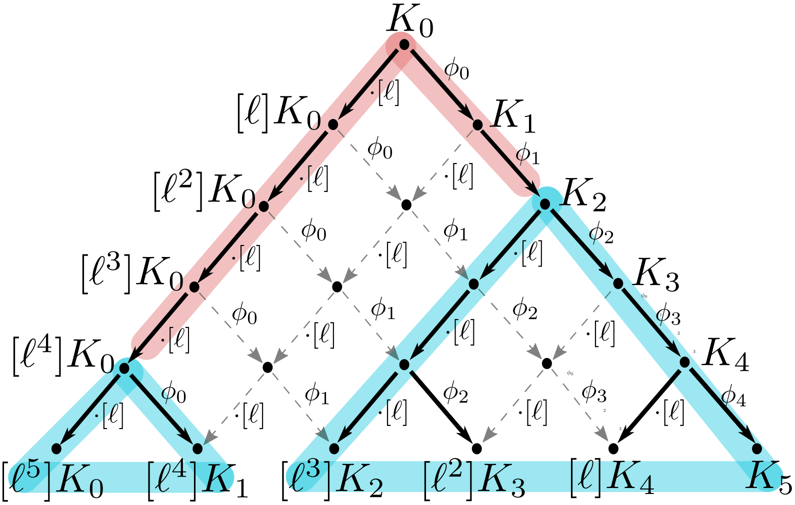

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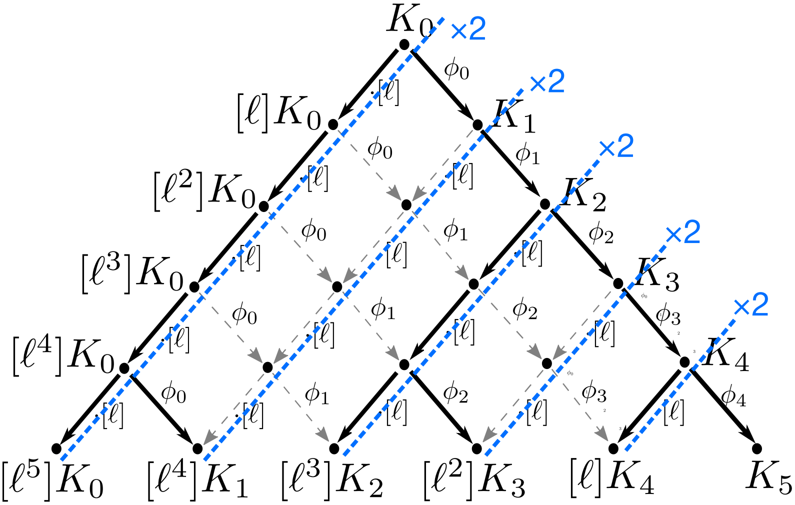

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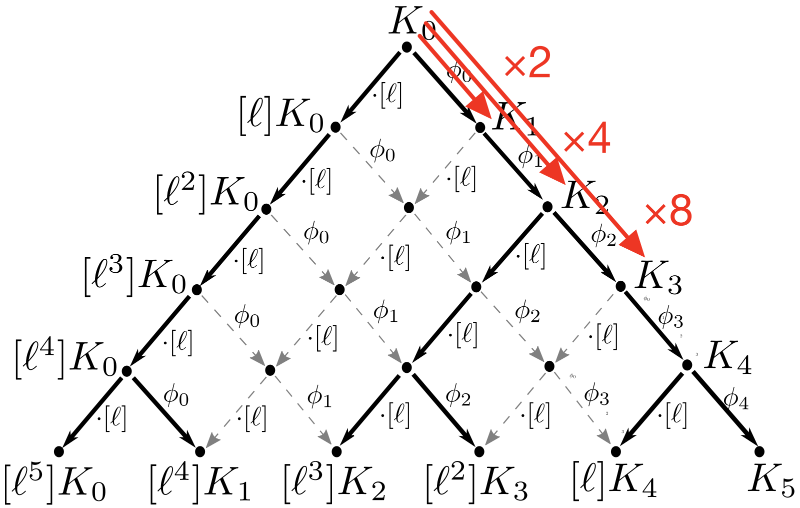

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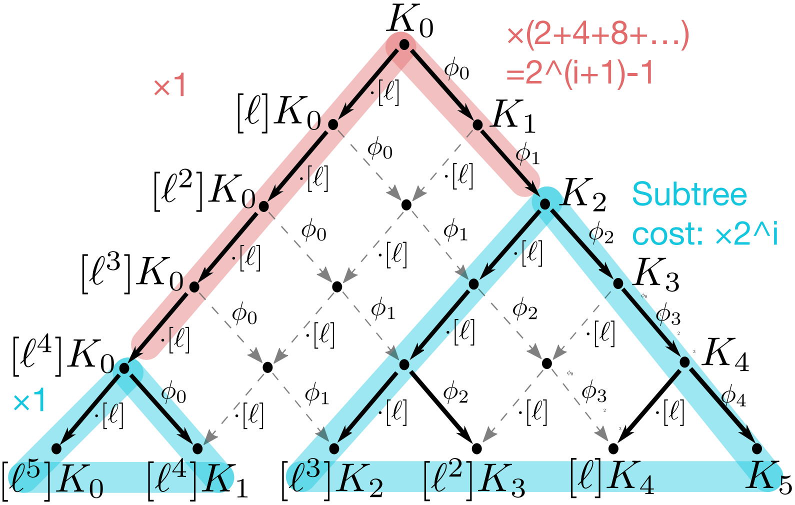

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Sort-and-Merge

- Standard MitM: hash-table (left tree), lookups (right tree)
  - $O(N)$ insertions/lookups
  - "Optimal" if $O(1)$ random memory access
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  - (Adj et al. 2019) considered 2D-mesh sorting as physically optimal. However, it is not clear at which scale the physical limits start to apply.
  - Although we use only 2^{45} storage, the limit of that paper set to 2^{80} seems not that far to actually reach physical limitations.

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   Example: two 2^{44}-sized trees, truncated to 64 bits
   $\Rightarrow \quad \approx 2^{44 \cdot 2 - 64} = 2^{24}$ collisions (false positives)

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In $IKEp182, at least $\times 5 - 6$ compression rate (vs 44-bit path + 64-bit j-invariant part)

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1. Introduction
2. Meet-in-the-Middle Isogeny Search
3. Computing a SIKE-tree
4. Intersecting two SIKE-trees
5. Application to $IKEp182
6. Conclusion

Application to \$IKEp182 (1/2)

\$IKEp182 challenge:

- $p = 2^{91}3^{57} - 1$ (182 bits); $e_A = 91, e_B = 57$
- 91 steps split: 45 (left tree) + 44 (right tree) + 2 (A leakage)
- Conjugation trick (Costello et al. 2020): left tree size 2^{44}
- MitM: compute and intersect two sets of 2^{44} j-invariants

Application to $IKEp182 (2/2)

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Total: 9.5 core-years and 256 TiB (70+ TiB should be enough)

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Conclusion

- Optimizations for MitM-based isogeny search
- Useful generic tool: sort-and-merge intersection of large sets of 64-bit elements
- The paper contains discussion on scalability of the approach

ia.cr/2021/1421 github.com/cryptolu/sike_mitm

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  4. (Castryck and Decru 2022) ≈ 1 laptop-weekend[1]

ia.cr/2021/1421 github.com/cryptolu/sike_mitm

[1] Actually, a few laptop-seconds, see optimized implementation by (Oudompheng and Pope 2022)

References I

Adj, Gora et al. (Aug. 2019). "On the Cost of Computing Isogenies Between Supersingular Elliptic Curves". In: *SAC 2018*. Ed. by Carlos Cid and Michael J. Jacobson Jr: vol. 11349. LNCS. Springer, Heidelberg, pp. 322–343. DOI: 10.1007/978-3-030-10970-7_15.

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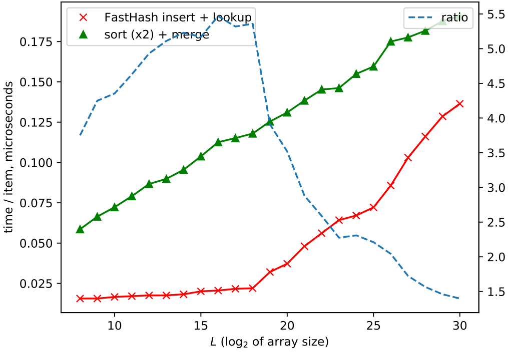
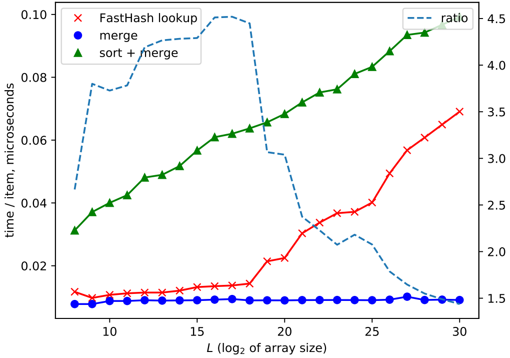
Comparison of HashTable and SortMerge on a PC

Performance comparison between FastHash and SortMerge
over 64-bit integer arrays of total size 2^L



Intersecting two arrays without precomputation



Lookup in an array with precomputation